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Lie algebroids, Lie groupoids and TFT

Francesco Bonechi[a,*], Maxim Zabzine[b,c]

[a] *I.N.F.N. and Dipartimento di Fisica, Via G. Sansone 1, 50019 Sesto Fiorentino - Firenze, Italy*
[b] *Department of Theoretical Physics, Uppsala University, Box 803, SE-75108 Uppsala, Sweden*
[c] *Kavli Institute of Theoretical Physics, University of California, Santa Barbara, CA 93106, USA*

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Abstract

We construct the moduli spaces associated to the solutions of equations of motion (modulo gauge transformations) of the Poisson sigma model with target being an integrable Poisson manifold. The construction can be easily extended to a case of a generic integrable Lie algebroid. Indeed for any Lie algebroid one can associate a BF-like topological field theory which localizes on the space of algebroid morphisms, that can be seen as a generalization of flat connections to the groupoid case. We discuss the finite gauge transformations and discuss the corresponding moduli spaces. We consider the theories both without and with boundaries.

Keywords: Groupoids; Algebroids; Moduli space on flat connections; Topological field theory

1. Introduction

Topological field theory (TFT) plays a prominent role in the investigation of geometry and topology of the moduli spaces of flat connections over a two-dimensional surface Σ. In particular these moduli spaces appear as the phase space of Chern-Simons theory, the localization locus of two-dimensional Yang-Mills and as the stationary points of BF theory. The application of quantum field theoretical methods has produced new results such as the formulas for the symplectic volume and for the intersection numbers (e.g., see [2]).

The Poisson sigma model (PSM) is another example of two-dimensional TFT introduced in [18,21] which is a sigma model defined on a two-dimensional surface Σ with target being a Poisson manifold. The BF-theory and A-model are particular examples of PSM. Recently PSM has attracted additional attention due to its relation with the deformation quantization [11]. However the potential use of the PSM as a nonperturbative TFT still remains to be investigated.

In this paper we show how a generalization of the moduli space of flat connections over Σ naturally appears when we study the stationary configurations of PSM. We complete the study of the moduli space of stationary points of PSM modulo the gauge transformations initiated in [5] for the special case of Poisson–Lie groups. In this situation the

* Corresponding author. Tel.: +39 055 4572305; fax: +39 055 4572364.
E-mail addresses: Francesco.Bonechi@fi.infn.it (F. Bonechi), Maxim.Zabzine@teorfys.uu.se (M. Zabzine).

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equations of motion of the model have a straightforward geometrical interpretation as the equations of a flat connection on the trivial bundle for the dual group and a parallel section for the fibre bundle associated to the dressing action. This observation dictates an obvious choice for the on-shell finite gauge transformations and the corresponding definition of the moduli space of solutions. Moreover the resulting space has a natural description where for every symplectic leaf of the target we associate the moduli space of flat connections for the isotropy group of the leaf.

In the present work we address the general case. Surprisingly it turns out that once we properly identify the gauge transformations the moduli space admits the same description. Furthermore the geometrical interpretation of the equations of PSM as algebroid morphisms between $T\Sigma$ and $T^*\mathcal{M}$ suggests that the discussion remains valid if we substitute $T^*\mathcal{M}$ by a generic (integrable) Lie algebroid E. This extension leads us beyond the PSM and puts the results in a different perspective.

An algebroid over a point is a Lie algebra and the moduli space of algebroid morphisms coincides with the moduli space of flat connections for the trivial bundle. Thus in the general case we can look at the equations of motion as a generalized flat connection equation where the structure constants depend on the base manifold. By applying the Lie theorems for integrable algebroids, we can equivalently deal with groupoid morphisms from the fundamental groupoid $\Pi(\Sigma)$ to the groupoid $\mathcal{G}(E)$ integrating E. This must be seen as the generalization of the holonomy description of the moduli space of flat G-connections as the space of G-representations of the fundamental group. Our result describes this generalized moduli space as the union over the leaves of the representations of the fundamental group in the isotropy group.

The structure of the paper is as follows. In Section 2 we review some basic notions from algebroid and groupoid theory. In Section 3 we discuss the equations of motion and the corresponding TFTs. Also we consider the natural boundary conditions in this setup. Section 4 is devoted to the finite gauge transformations which form a groupoid. We explain their relation to the algebroid (groupoid) homotopy. In Section 5 we discuss the moduli spaces and their equivalence to the various generalizations of flat connections. Section 6 contains the summary and the list of open problems.

2. Lie algebroids and Lie groupoids

In this section we recall some basic notions from the theory of Lie algebroids and Lie groupoids. For a more extensive discussion we refer to [9] and [20].

2.1. Lie algebroids

Definition 2.1. A *Lie algebroid* $(E, \mathcal{M}, \rho, \{\ ,\ \})$ is a vector bundle E over a manifold $\mathcal{M}$ together with a bundle map (the anchor) $\rho : E \to T\mathcal{M}$ and a Lie bracket $\{\ ,\ \}$ on the space $\Gamma(E)$ of sections of E satisfying the compatibility condition

$$\{v, fu\} = f\{v, u\} + \mathcal{L}_{\rho_* v} f u, \quad u, v \in \Gamma(E), f \in C^\infty(\mathcal{M}) \tag{2.1}$$

where $\rho_* : \Gamma(E) \to \Gamma(T\mathcal{M})$ is the induced map of sections and $\mathcal{L}$ is the Lie derivative.

It follows from the definition that ρ_* is a morphism of Lie algebras. On a trivializing chart U we can choose the local coordinates X^μ $(\mu = 1, \ldots, \dim \mathcal{M})$ and a basis e^A, $(A = 1, \ldots, \mathrm{rank} E)$ on the fiber (e.g., the basis of constant sections on $E|_U$). In these local coordinates we introduce the anchor $\rho^{\mu A}$ and the structure functions

$$\rho(e^A)(X) = \rho^{\mu A}(X)\partial_\mu, \qquad \{e^A, e^B\} = f^{AB}{}_C e^C . \tag{2.2}$$

The compatibility condition (2.1) implies the following equations

$$\rho^{\nu A}\partial_\nu \rho^{\mu B} - \rho^{\nu B}\partial_\nu \rho^{\mu A} = f^{AB}{}_C \rho^{\mu C} \tag{2.3}$$

$$\rho^{\mu[D}\partial_\mu f^{AB]}{}_C + f^{[AB}{}_L f^{D]L}{}_C = 0, \tag{2.4}$$

where [ ] stands for the antisymmetrization.

To any Lie algebroid we associate a characteristic foliation, that is the singular foliation on $\mathcal{M}$ determined by the distribution $x \to \mathrm{Im}(\rho|_{E_x})$. The isotropy algebra for $x \in \mathcal{M}$ is defined as the kernel of the anchor map ρ

$$\mathbf{g}_x = \ker(\rho|_{E_x}). \tag{2.5}$$

For any x, y in the same leaf $\in L$ we have $\mathbf{g}_x \simeq \mathbf{g}_y$. Hence we get a bundle of Lie algebras over L

$$\mathbf{g}_L = \bigcup_{x \in L} \mathbf{g}_x \to L. \tag{2.6}$$

Here are some examples of Lie algebroids which will be relevant for further discussion.

Example 2.2. Every Lie algebra $\mathbf{g}$ is an algebroid over a point ($\rho = 0$).

Example 2.3. The tangent bundle $T\mathcal{M}$ of a smooth manifold $\mathcal{M}$ is an algebroid with the bracket between vector fields and $\rho = \mathrm{id}$.

Example 2.4. Let $\gamma : \mathbf{g} \to \mathrm{Vect}(\mathcal{M})$ be a right action of a Lie algebra $\mathbf{g}$ on a manifold $\mathcal{M}$. The *action Lie algebroid* is defined on $\mathcal{M} \times \mathbf{g}$ over $\mathcal{M}$ with anchor $\rho(m, \xi) = \gamma(\xi)(m) \in T_m\mathcal{M}$ and bracket between $v, w \in \Gamma(\mathcal{M} \times \mathbf{g}) = C^\infty(\mathcal{M}, \mathbf{g})$

$$\{v, w\}(m) = [v(m), w(m)] + \mathcal{L}_{\gamma(v(m))}(w)(m) - \mathcal{L}_{\gamma(w(m))}(v)(m),$$

where [,] denotes the bracket in $\mathbf{g}$ and $\mathcal{L}$ the Lie derivative.

Example 2.5. Let $\mathcal{M}$ be a Poisson manifold with Poisson tensor $\alpha \in \Gamma(\wedge^2 T\mathcal{M})$. The associated canonical algebroid is defined on $T^*\mathcal{M}$ by choosing as anchor the contraction $\sharp_\alpha$ of cotangent vectors with α and by defining the bracket on exact forms as $\{df, dg\} = d\{f, g\}$, $f, g \in C^\infty(\mathcal{M})$ and extending it to all $\Gamma(T^*\mathcal{M})$ with (2.1).

Next following Higgins and Mackenzie [16] we give a definition of Lie algebroid morphism which plays a central role in our investigation:

Definition 2.6. Let $(E_1, \mathcal{M}_1, \rho_1, \{\ ,\ \}_1)$ and $(E_2, \mathcal{M}_2, \rho_2, \{\ ,\ \}_2)$ be Lie algebroids. Then *a morphism of Lie algebroids* is a vector bundle morphism

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ \mathcal{M}_1 & \xrightarrow{\phi} & \mathcal{M}_2 \end{array} \tag{2.7}$$

such that

$$\rho_2 \circ \Phi = d\phi \circ \rho_1, \tag{2.8}$$

where $d\phi : T\mathcal{M}_1 \to T\mathcal{M}_2$ and such that for arbitrary $V, W \in \Gamma(E_1)$ with Φ-decomposition

$$\Phi \circ V = \sum V^i (e_i \circ \phi), \qquad \Phi \circ W = \sum W^i (\tilde{e}_i \circ \phi) \tag{2.9}$$

where $W^i, V^i \in C^\infty(\mathcal{M}_1)$ and $e_i, \tilde{e}_i \in \Gamma(E_2)$, we have

$$\Phi \circ \{V, W\}_1 = \sum V^i W^j (\{e_i, \tilde{e}_j\}_2 \circ \phi) + \sum \mathcal{L}_{\rho_{*1} V} W^i (\tilde{e}_i \circ \phi) - \sum \mathcal{L}_{\rho_{*1} W} V^i (e_i \circ \phi). \tag{2.10}$$

It is clear that relations (2.9) and (2.10) are in $\phi^* E_2$. It may appear that there are ambiguities in this definition. However it can be shown that the right-hand side of (2.10) is independent of the Φ-decompositions of V and W, for further details see [16].

Definition 2.7. Let $(E, \mathcal{M}, \rho, \{\ ,\ \})$ be a Lie algebroid. Then a *Lie subalgebroid* of E is a morphism of Lie algebroids $\Phi : E' \to E$, $\phi : \mathcal{D} \to \mathcal{M}$ such that Φ and ϕ are injective immersions.

In local coordinates we can describe a Lie subalgebroid as follows. In the neighborhood of a point $x \in \mathcal{D}$ (we identify $\mathcal{D}$ with a submanifold of $\mathcal{M}$) we choose coordinates $X^\mu = (X^{\hat{\mu}}, X^{\tilde{\mu}})$ adapted to the submanifold $\mathcal{D}$ such that in this neighborhood the submanifold is given by the condition $X^{\hat{\mu}} = 0$. We use the Greek lower case letters with a hat for the coordinates transverse to the submanifold $\mathcal{D}$ and the same letters with tilde for the coordinates along the submanifold $\mathcal{D}$. We can as well introduce the basis on the fiber adapted to the fact that $E'_x \subset E_x$, namely $e^A = (e^a, e^n)$. We use the Latin lower case letters from the beginning of alphabet for the basis of E'_x and from the middle for the remaining elements in the basis. Then one can show that the above definition implies the following properties for the anchor map and for "structure constants" along $\mathcal{D}$

$$\rho^{\hat{\mu}a}(0, X^{\tilde{\mu}}) = 0, \qquad f^{ab}{}_n(0, X^{\tilde{\mu}}) = 0. \tag{2.11}$$

Thus $\rho^{\tilde{\mu}a}(0, X^{\tilde{\mu}})$ and $f^{ab}{}_c(0, X^{\tilde{\mu}})$ define the structure of a Lie algebroid over $E' \to \mathcal{D}$.

2.2. Lie groupoids

A *groupoid* is a small category $\mathcal{G}$ with all arrows invertible. If the set of objects (points) is $\mathcal{M}$, we say that $\mathcal{G}$ is a groupoid over $\mathcal{M}$. We shall denote by the same letter $\mathcal{G}$ the space of arrows, and write

$$\begin{array}{c}\mathcal{G}\\ s\big\downarrow\big\downarrow t\\ \mathcal{M}\end{array}$$

where s and t are the source and target maps. If $g, h \in \mathcal{G}$ the product gh is defined only for pairs (g, h) in the set of composable arrows

$$\mathcal{G}^{(2)} = \{(g, h) \in \mathcal{G} \times \mathcal{G} | t(h) = s(g)\},$$

and we denote by $g^{-1} \in \mathcal{G}$ the inverse of g, and by $\mathrm{id}(x) \equiv x$ the identity arrow at $x \in \mathcal{M}$. The objects $\mathcal{M}$ are thus embedded in $\mathcal{G}$ with id; when no confusion arises we will omit id and simply consider $\mathcal{M} \subset \mathcal{G}$. If $\mathcal{G}$ and $\mathcal{M}$ are topological spaces, all the maps are continuous, and s and t are open surjections, we say that $\mathcal{G}$ is a *topological groupoid*. A *Lie groupoid* is a groupoid where the space of arrows $\mathcal{G}$ and the space of objects $\mathcal{M}$ are smooth manifolds, the source and target maps s, t are submersions, and all the other structure maps are smooth. We require $\mathcal{M}$ and the s-fibers $\mathcal{G}_x = s^{-1}(x)$, where $x \in \mathcal{M}$, to be Hausdorff manifolds, but it is important to allow the total space $\mathcal{G}$ of arrows to be non-Hausdorff.

The action of $\mathcal{G}$ on a space X equipped with an anchor $\mu : X \to \mathcal{M}$ consists in a map from $\mathcal{G} * X = \{(g, x) \in \mathcal{G} \times X \mid s(g) = \mu(x)\}$ to X, $(g, x) \to gx$ such that: (i) $\mu(gx) = t(g)$, (ii) $g(hx) = (gh)x$, (iii) $\mu(x)x = x$.

Given a Lie groupoid $\mathcal{G}$ we can define the Lie algebroid $\mathcal{A}(\mathcal{G})$ as follows. It is defined on $\mathcal{A}(\mathcal{G})_x = T_x\mathcal{G}_x$, for $x \in \mathcal{M}$; the anchor is $\rho = \mathrm{d}t : T_x\mathcal{G}_x \to T_x\mathcal{M}$. The bracket comes from the identification of $\Gamma(\mathcal{A}(\mathcal{G}))$ with left invariant vector fields on $\mathcal{G}$ by choosing the bracket of vector fields on $\mathcal{G}$. Since not every Lie algebroid comes out in this way, we say that an algebroid E is *integrable* if there exists a Lie groupoid $\mathcal{G}$ such that $\mathcal{A}(\mathcal{G}) = E$. The problem of integration of Lie algebroid is a generalization of the problem of integration of Lie algebras.

A groupoid morphism from a groupoid $\mathcal{G}_1$ to $\mathcal{G}_2$ is a covariant functor; more explicitly, we get the following definition:

Definition 2.8. Let $\mathcal{G}_i$, $i = 1, 2$, be Lie groupoids and let $\mathcal{M}_i, \mathrm{id}_i, s_i, t_i$ be the corresponding space of units, their embedding, the source and target maps. A groupoid morphism from $\mathcal{G}_1$ to $\mathcal{G}_2$ is a couple of maps $(X, \hat{X})$, $X : \mathcal{M}_1 \to \mathcal{M}_2$ and $\hat{X} : \mathcal{G}_1 \to \mathcal{G}_2$ such that

(i) $X \circ s_1 = s_2 \circ \hat{X}$, $X \circ t_1 = t_2 \circ \hat{X}$;
(ii) $\hat{X}(ab) = \hat{X}(a)\hat{X}(b)$ for all $a, b \in \mathcal{G}_1^{(2)}$;
(iii) $\hat{X} \circ \mathrm{id}_1 = \mathrm{id}_2 \circ X$.

Let $E_i = \bigcup_{x \in \mathcal{M}_i} T_{\mathrm{id}_i(x)}(\mathcal{G}_i)_x$ be the tangent Lie algebroids. It is a fundamental fact that (X, j), where $j = \hat{X}_* : (E_1)_x \to (E_2)_{X(x)}$ is a Lie algebroid morphism.

We have the following Lie theorems for algebroids.

Theorem 2.9 (*Lie I*). *Let $E = \mathcal{A}(\mathcal{G})$ and let $E' \subset E$ be a subalgebroid. Then there exists an immersed Lie subgroupoid $\mathcal{G}' \subset \mathcal{G}$ such that $E' = \mathcal{A}(\mathcal{G}')$.*

Theorem 2.10 (*Lie II*). *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two Lie groupoids with Lie algebroids E_1 and E_2; if $\mathcal{G}_1$ is source simply connected (ssc) then for every Lie algebroid morphism $(X, j) : E_1 \to E_2$ there exists a unique groupoid morphism $(X, \hat{X}) : \mathcal{G}_1 \to \mathcal{G}_2$, such that $j = \hat{X}_*$.*

Moreover for a given integrable Lie algebroid E there exists a unique source simply connected Lie groupoid $\mathcal{G}(E)$ integrating it. Next we give some basic examples of Lie groupoids.

Example 2.11. A finite dimensional Lie algebra $\mathbf{g}$, considered as a Lie algebroid over a point as in Example 2.2, is integrated by the simply connected group G seen as a groupoid over a point.

Example 2.12. The source simply connected groupoid integrating $T\mathcal{M}$ (see Example 2.3) is the fundamental groupoid $\Pi(\mathcal{M})$, the set of curves in $\mathcal{M}$ modulo homotopies with fixed end points; the groupoid structures are the obvious ones, e.g. source (resp. target) is the initial (resp. final) point, multiplication is concatenation, and identities are the trivial loops. For each $m \in \mathcal{M}$, $\Pi(\mathcal{M})_m$ is diffeomorphic to the universal cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$ and $\Pi(\mathcal{M})_m^m$ is $\pi_1(\mathcal{M}, m)$.

Example 2.13. If the action of $\mathbf{g}$ on $\mathcal{M}$ of Example 2.4 comes from an action of G, where $\mathbf{g} = \text{Lie}G$, then the action Lie algebroid on $\mathcal{M} \times \mathbf{g}$ is integrated by the action Lie groupoid $\mathcal{M} \times G$.

Example 2.14. If the algebroid $T^*\mathcal{M}$ associated to a Poisson manifold (see Example 2.5) is integrable, then the groupoid is a symplectic manifold, called the *symplectic groupoid*. The particular case of a Poisson Lie group is always integrable. In the factorizable case, the groupoid integrating it corresponds to the action groupoid $G \times G^*$, where G^* is the dual Poisson–Lie group, acting on G with the dressing transformations, see [19].

We close this section by defining the admissible sections of a groupoid. For any Lie groupoid $\mathcal{G}$ the group of *admissible sections* $\text{Bis}(\mathcal{G})$ is the group of maps $\sigma : \mathcal{M} \to \mathcal{G}$, such that $s\sigma = \text{id}$ and $t\sigma = \psi_\sigma : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism. The group law is $\sigma_1\sigma_2(x) = \sigma_1(t(\sigma_2(x)))\sigma_2(x)$ and the identity is id. Since we have that $\psi_{\sigma_1\sigma_2} = \psi_{\sigma_1}\psi_{\sigma_2}$, ψ_σ defines an action of $\text{Bis}(\mathcal{G})$ on $\mathcal{M}$ that preserves the leaves. It comes out that $\text{Bis}(\mathcal{G}(E))$ is a Lie group whose Lie algebra is $\Gamma(E)$. Indeed a tangent vector to $\sigma \in \text{Bis}(\mathcal{G}(E))$ means to assign an element of $T_{\sigma(x)}\mathcal{G}_x$ for each $x \in \mathcal{M}$; in particular the tangent space to the identity $\sigma = \text{id}$ is $\Gamma(E)$.

3. Lie algebroid and TFT

To any Lie algebroid $(E, \mathcal{M}, \rho, \{\ ,\ \})$ we can associate a gauge theory in the following way. Consider the space of bundle maps from the tangent bundle $T\Sigma$ of a two-dimensional oriented manifold Σ, possibly with boundary, to the vector bundle E with base manifold $\mathcal{M}$. We describe such bundle maps by a pair (X, j),

$$\begin{array}{ccc} T\Sigma & \xrightarrow{j} & E \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{X} & \mathcal{M} \end{array} \tag{3.12}$$

where $X : \Sigma \to \mathcal{M}$ is the base map and j is the map between fibers, e.g. j is a section in $\Gamma(T^*\Sigma \otimes X^*E) = \Omega^1(\Sigma, X^*E)$. Now we consider on $T\Sigma$ the tangent algebroid and we require that (X, j) is a Lie algebroid morphism. In local coordinates $\{X^\mu\}$ on $\mathcal{M}$ and $\{u^\alpha\}$ on Σ and by choosing a local trivialization e^A for E, X is given by $(\dim \mathcal{M})$ functions $X^\mu(u)$ and j by $(\text{rank}E)$ differential 1-forms $j_A = j_{A\alpha}du^\alpha$. For arbitrary vector fields on Σ, $V, W \in \Gamma(T\Sigma)$ we get the j-decomposition

$$j \circ V = j_{A\alpha}V^\alpha(e^A \circ X), \qquad j \circ W = j_{A\alpha}W^\alpha(e^A \circ X). \tag{3.13}$$

Applying the Definition 2.6, we can write (2.10) in local coordinates

$$j_{A\alpha}[V, W]^\alpha(e^A \circ X) = j_{A\alpha}V^\alpha j_{B\beta}W^\beta(\{e^A, e^B\} \circ X) + V^\alpha\partial_\alpha(j_{A\beta}W^\beta)(e^A \circ X) - W^\alpha\partial_\alpha(j_{A\beta}V^\beta)(e^A \circ X) \tag{3.14}$$

where [,] is the standard Lie bracket on $\Gamma(T\Sigma)$ and { , } is the bracket on $\Gamma(E)$. The Eq. (3.14) implies

$$V^\alpha W^\beta(\partial_\beta j_{A\alpha} - \partial_\alpha j_{A\beta} - f^{BC}{}_A j_{B\alpha} j_{C\beta})(e^A \circ X) = 0 \tag{3.15}$$

where we have used (2.2). To summarize, the equations of motion of the system are given by

$$\mathrm{d}j_A + \frac{1}{2} f^{BC}{}_A(X) j_B \wedge j_C = 0, \tag{3.16}$$

$$\mathrm{d}X^\mu - \rho^{\mu A}(X) j_A = 0, \tag{3.17}$$

where the last equation is a simple consequence of (2.8).

Thus the fact that (X, j) is a Lie algebroid morphism implies the first order differential equations for X^μ and $j_{A\alpha}\mathrm{d}\xi^\alpha$. These equations form a consistent system of partial differential equations due to the properties (2.3) and (2.4). On top of this the system is invariant under the infinitesimal gauge transformations

$$\delta j_A = -\mathrm{d}\beta^A - f^{BC}{}_A(X) j_B \beta_C \tag{3.18}$$

$$\delta X^\mu = -\rho^{\mu A}(X)\beta_A \tag{3.19}$$

where β is a gauge parameter. For the similar discussion of the system (3.16)–(3.19) see [3].

The motivating example for the system (3.16)–(3.19) is PSM where Σ is two-dimensional and $E = T^*\mathcal{M}$ defined in Example 2.5. In this case the Eq. (3.16), (3.17) are the stationary points of the action functional

$$S(X, \eta) = \int_\Sigma \langle \eta, \mathrm{d}X \rangle + \frac{1}{2}\langle \eta, (\alpha \circ X)\eta \rangle, \tag{3.20}$$

where (X, η) is bundle morphism from $T\Sigma$ to $T^*\mathcal{M}$. The pairing $\langle , \rangle$ is defined as pairing of the values in $T\mathcal{M}$ and $T^*\mathcal{M}$ and the exterior product of differential forms. The action functional is invariant under the corresponding transformations (3.18), (3.19).

Indeed the system (3.16)–(3.19) is defined for Σ being a manifolds of any dimension, $\dim \Sigma = p$. The equations of motion can be derived from the following trivial action

$$S(X, j, A, \lambda) = \int_\Sigma \lambda^A \wedge \left(\mathrm{d}j_A + \frac{1}{2} f^{BC}{}_A(X) j_B \wedge j_C \right) + A_\mu \wedge (\mathrm{d}X^\mu - \rho^{\mu A}(X) j_A), \tag{3.21}$$

where $\lambda \in \Omega^{p-2}(\Sigma, X^*E^*)$ and $A \in \Omega^{p-1}(\Sigma, X^*T^*\mathcal{M})$ are the Lagrangian multipliers, see [22] for an analogous discussion. The action (3.21) is invariant under (3.18) and (3.19) together with the additional transformations

$$\delta\lambda^A = \rho^{\mu A} b_\mu + f^{AB}{}_C \lambda^C \beta_B \tag{3.22}$$

$$\delta A_\mu = (-1)^{(p-1)}\mathrm{d}b_\mu - (\partial_\mu f^{AB}{}_C)\lambda^C \wedge j_A \beta_A - (\partial_\mu \rho^{\nu A}) b_\nu \wedge j_A + (\partial_\mu \rho^{\nu A}) A_\nu \beta_A, \tag{3.23}$$

where $b \in \Omega^{p-2}(\Sigma, X^*T^*M)$ and $\beta \in \Omega^0(X^*E)$ are the gauge parameters. In coordinate free way the fields (X, j, A, λ) can be interpreted as follows. (X, j) is a bundle morphism $T\Sigma \to E$, (X, A) is a bundle morphism $\wedge^{p-1} T\Sigma \to T^*\mathcal{M}$ and (X, λ) is a bundle morphism $\wedge^{p-2} T\Sigma \to E^*$. Introducing the pairings as pairing in values between E and E^* and as paring between $T\mathcal{M}$ and $T^*\mathcal{M}$ and the exterior product of differential forms on Σ we can rewrite the action functional (3.21) in coordinate independent form. This theory is a rather obvious generalization of *BF*-theory [17,1] to the case of a generic Lie algebroid.

If Σ is two-dimensional then the action (3.21) can be written in the following form

$$S = \int_\Sigma j_A \wedge \mathrm{d}\lambda^A + A_\mu \wedge \mathrm{d}X^\mu + \frac{1}{2}(f^{BC}{}_A(X)\lambda^A) j_B \wedge j_C + \rho^{\mu A}(X) j_A \wedge A_\mu \tag{3.24}$$

and it differs from (3.21) only by a boundary term. The action (3.24) has a clear interpretation, this is the Poisson sigma model for E^*. In fact, if $(E, \mathcal{M}, \rho, \{ , \})$ is a Lie algebroid, the dual bundle E^* has a natural Poisson structure, defined by the tensor $\pi \in \wedge^2 T^*E^*$ given in coordinates X^μ, λ^A as

$$\pi(X, \lambda) = f^{AB}{}_C \lambda^C \frac{\partial}{\partial\lambda^A} \wedge \frac{\partial}{\partial\lambda^B} + \rho^{\mu A} \frac{\partial}{\partial\lambda^A} \wedge \frac{\partial}{\partial X^\mu}. \tag{3.25}$$

In the action (3.24) we have that $(X, \lambda) : \Sigma \to E^*$ and (A, j) is a differential form on Σ taking values in the pull-back by (X, λ) of T^*E^*.

Every solution of (3.16) and (3.17) defines a solution of the equations of motion of (3.24), e.g. we have embedded our system of equations in a TFT. Let us see it in an intrinsic way. The full set of equations, including those obtained by varying X and j, obviously describes the algebroid morphisms from $T\Sigma$ to T^*E^*. In Lemma 4.2 of [10] it is shown that E is a subalgebroid of T^*E^*. The injection is defined as follows: the fibre $T^*_{(m,\alpha)}E^*$ over $(m, \alpha) \in \mathcal{M} \times E_m$, is $T^*_m\mathcal{M} \oplus E_m$. The injection is defined as $\iota : E_m \to T^*_{(m,0)}E^*$ as $\iota(m, a) = ((m, 0), 0 \oplus a)$. It comes out that this is an injective algebroid morphism. Thus composing with ι we can inject the set of vector bundle morphisms from $T\Sigma$ to E into the space of fields of the model and every algebroid morphism of E defines an algebroid morphism for T^*E^*, e.g. is a solution of the equations of motion of the PSM with target E^*. Of course this mapping is not surjective, e.g. there are solutions with λ and A different from 0.

If $\partial\Sigma \neq \emptyset$ we have to choose appropriate boundary conditions on the fields. For example, we may ask that the boundary terms vanish in the variations of (3.24). Thus in order to get the equations of motion, we have to choose boundary conditions such that

$$(j_{\tau A}\delta\lambda^A + A_{\tau\mu}\delta X^\mu)|_{\partial\Sigma} = 0 \tag{3.26}$$

where $j_A|_{T^*\partial\Sigma} = j_{A\tau}\mathrm{d}\tau$ and $A_\mu|_{T^*\partial\Sigma} = A_{\mu\tau}\mathrm{d}\tau$. The action (3.24) is invariant under the infinitesimal gauge transformations (3.18), (3.19) and (3.22), (3.23) with parameter $(b_\mu \mathrm{d}X^\mu, \beta_A \mathrm{d}\lambda^A) \in \Gamma((X, \lambda)^*(T^*E^*))$, provided the following boundary condition is satisfied

$$(\beta_A\partial_\tau\lambda^A + b_\mu\partial_\tau X^\mu)|_{\partial\Sigma} = 0. \tag{3.27}$$

Finally, also the boundary conditions should be invariant under the residual gauge transformations. By using the results of [13] (see also [5]) one can establish that the boundary conditions for the theory are labeled by the co-isotropic submanifolds of E^*. Recall that a submanifold X of a Poisson manifold E^* is co-isotropic iff the co-normal bundle N^*X is a subalgebroid of T^*E^*.

Motivated by this discussion it is natural to choose those boundary conditions that come from E. In fact for every subalgebroid $E' \subset E$ over $\mathcal{D} \subset \mathcal{M}$, we can see that $E'^\perp \subset E^*$ is a co-isotropic submanifold. Let us choose the adapted coordinates $X^\mu = (X^{\hat\mu}, X^{\tilde\mu})$ and trivialization $e_A = (e_a, e_n)$ such that the $\mathcal{D}$ corresponds to $X^{\hat\mu} = 0$ and $E'_X = \langle e_a\rangle$. It is then easy to verify that the local conditions (2.11) for E' is a subalgebroid correspond to those for $E'^\perp$, a co-isotropic submanifold, $\pi^{ab} = 0$ and $\pi^{a\hat\mu} = 0$. The contrary is not true as can be easily understood from the following example.

Example 3.1. Let $E = \mathbf{g}$ be a Lie algebra and $\mathbf{g}^*$ is vector space with the canonical Poisson structure. Then the subalgebroids of $\mathbf{g}$ are the Lie subalgebras $\mathbf{h} \subset \mathbf{g}$ that define the linear co-isotropic submanifolds $\mathbf{h}^\perp \subset \mathbf{g}^*$. However not any co-isotropic submanifold of $\mathbf{g}^*$ arises in this way.

Thus in the forthcoming discussion when we refer to the open case we consider the system (3.16)–(3.19) with the boundary conditions given by Lie subalgebroids of $(E, \mathcal{M}, \rho, \{\ ,\ \})$

$$\begin{array}{ccccc} T\partial\Sigma & \xrightarrow{j} & E' & \xrightarrow{\Phi} & E \\ \downarrow & & \downarrow & & \downarrow \\ \partial\Sigma & \xrightarrow{X} & \mathcal{D} & \xrightarrow{\phi} & \mathcal{M} \end{array} \tag{3.28}$$

On the boundary the gauge transformations are restricted correspondingly.

We close this section with a comment about PSM with target $\mathcal{M}$ when $\mathcal{M}$ is a Poisson manifold. The boundary conditions for this PSM are not defined by any subalgebroid of $T^*\mathcal{M}$, but only by those which are co-normal bundles of a submanifold [13]. In [8] a more general class of boundary conditions is considered. This is not surprising since we have motivated our choice of boundary conditions starting from PSM with target $(T^*\mathcal{M})^*$. It is not clear at the moment the relevance of this wider class of boundary conditions in the context of the PSM with target $\mathcal{M}$.

4. Integration of gauge transformations

In this section we define the finite gauge transformations that integrate the infinitesimal transformations (3.18) and (3.19). In the case of an integrable Lie algebroid, we will analyze groupoid morphisms rather than algebroid morphisms, since it is much easier to introduce the finite gauge transformations.

In fact, due to Theorem 2.10 every solution (X, j) of the Eqs. (3.16) and (3.17) can be lifted to a groupoid morphism $(X, \hat{X})$ between $\Pi(\Sigma)$ and $\mathcal{G}(E)$, the (ssc) groupoid integrating E, and vice-versa. In the following we will identify the solutions of the Eqs. (3.16) and (3.17) with the groupoid morphisms from $\Pi(\Sigma)$ to $\mathcal{G}(E)$ that they generate. We denote the space of all morphisms from $\Pi(\Sigma)$ to $\mathcal{G}(E)$ with $\mathrm{Mor}(\Pi(\Sigma), \mathcal{G}(E))$.

4.1. The closed case

In this subsection we consider the case when $\partial\Sigma = \emptyset$. We assume that E is integrated by $\mathcal{G}(E)$ with $s, t : \mathcal{G}(E) \to \mathcal{M}$ being the source and target map. Let $\mathrm{id} : \mathcal{M} \to \mathcal{G}$ be the usual embedding of $\mathcal{M}$ in $\mathcal{G}(E)$ as the space of identities. As usual we denote with $\mathcal{G}(E)_x$ ($\mathcal{G}(E)^x$) the fiber of the source (target) map in $x \in \mathcal{M}$ and with $\mathcal{G}(E)_x^y = \mathcal{G}(E)_x \cap \mathcal{G}(E)^y$. Recall that $\mathcal{G}(E)_x$ and $\mathcal{G}(E)^x$ are separable smooth manifolds and that $T_x\mathcal{G}(E)_x = E_x$.

We will first define the gauge transformations on the morphisms from $\Pi(\Sigma)$ to $\mathcal{G}(E)$ and then we will compute the induced transformations on the algebroid morphisms between $T\Sigma$ and E.

Following [4] we introduce the infinite-dimensional groupoid $\mathcal{G}^\Sigma = \{\hat{\Phi}\colon \Sigma \to \mathcal{G}(E)\}$ over $\mathcal{M}^\Sigma = \{\Phi\colon \Sigma \to \mathcal{M}\}$ with structure maps defined pointwise. Namely, we define source and target by $s(\hat{\Phi})(u) = s(\hat{\Phi}(u))$, $t(\hat{\Phi})(u) = t(\hat{\Phi}(u))$ for $u \in \Sigma$ and multiplication by $\hat{\Phi}_1\hat{\Phi}_2(u) = \hat{\Phi}_1(u)\hat{\Phi}_2(u)$. A section S of the associated algebroid[1] $A(\mathcal{G}^\Sigma)$ is defined by giving a section $S(\Phi) \in \Gamma(\Phi^*E)$ for every $\Phi \in \mathcal{M}^\Sigma$. There is a natural groupoid action of $\mathcal{G}^\Sigma$ on $\mathrm{Mor}(\Pi(\Sigma), \mathcal{G}(E))$ which is given by

$$X_{\hat{\Phi}}(u) = t(\hat{\Phi})(u) \qquad \hat{X}_{\hat{\Phi}}([c_{uv}]) = \hat{\Phi}(u)\hat{X}([c_{uv}])\hat{\Phi}(v)^{-1}, \tag{4.29}$$

where $(X, \hat{X}), (X_{\hat{\Phi}}, \hat{X}_{\hat{\Phi}}) \in \mathrm{Mor}(\Pi(\Sigma), \mathcal{G}(E))$, $\hat{\Phi} \in \mathcal{G}^\Sigma$ with $s(\hat{\Phi}) = X$ and $[c_{uv}]$ is the homotopy class of a curve c_{uv} in Σ. Thus we declare $\mathcal{G}^\Sigma$ as our choice of finite gauge transformations.

However there are alternative choices of finite gauge transformations, e.g. the group $\mathrm{Bis}(\mathcal{G}^\Sigma)$ of admissible sections of $\mathcal{G}^\Sigma$. In this case the formula (4.29) also defines a group action of $\mathrm{Bis}(\mathcal{G}^\Sigma)$ on $\mathrm{Mor}(\Pi(\Sigma), \mathcal{G}(E))$. The orbits of $\mathcal{G}^\Sigma$ contain the orbits of $\mathrm{Bis}(\mathcal{G}^\Sigma)$ and thus the choice of $\mathcal{G}^\Sigma$ is a more generic one. Another possible choice of the gauge transformations is $(\mathrm{Bis}\mathcal{G}(E))^\Sigma$, which is the group of maps from Σ to $\mathrm{Bis}\mathcal{G}(E)$, seen as a subgroup of $\mathrm{Bis}(\mathcal{G}^\Sigma)$, see section 3.1 in [4]. However it is very hard to work with these groups and in the following we will consider only the groupoid action of $\mathcal{G}^\Sigma$ on $\mathrm{Mor}(\Pi(\Sigma), \mathcal{G}(E))$.

Indeed the choice of $\mathcal{G}^\Sigma$ as finite gauge transformations looks natural from the categorical point of view. Namely, if we look to the above groupoids as categories then any groupoid morphism in $\mathrm{Mor}(\Pi(\Sigma), \mathcal{G}(E))$ is a covariant functor from $\Pi(\Sigma)$ to $\mathcal{G}(E)$ and a gauge transformation between two groupoid morphisms as defined in (4.29) is a natural transformation between the functors. In Section 6 we will comment more on this issue.

There is another possibility to introduce the notion of gauge equivalence between the groupoid morphisms or algebroid morphisms, e.g. see [3]. Namely, this can be done via groupoid (algebroid) homotopies. The groupoid (algebroid) homotopy is an alternative way of integrating the gauge transformations (3.18) and (3.19). Let $I = [0, 1]$; it is clear that for groupoid $\Pi(\Sigma)$ over Σ, we can define on $\Pi(\Sigma) \times I \times I$ a groupoid structure over $\Sigma \times I$ with the corresponding algebroid given by $T(\Sigma \times I)$.

Definition 4.1. Let $\hat{X}_i$, $i = 1, 2$, be two groupoid morphisms from $\Pi(\Sigma)$ to $\mathcal{G}(E)$. We say that $\hat{X}_1$ and $\hat{X}_2$ are *homotopic* if there exists a groupoid morphism $\hat{X}_{12} : \Pi(\Sigma) \times I \times I \to \mathcal{G}(E)$ such that $\hat{X}_{12}(-, 0, 0) = \hat{X}_1$ and $\hat{X}_{12}(-, 1, 1) = \hat{X}_2$.

[1] This algebroid has been defined for one dimensional Σ in [6]. It has been done intrinsically in terms of the Lie algebroid E and thus it exists also for nonintegrable algebroids.

Definition 4.2. Let (X_i, j_i), $i = 1, 2$, be two algebroid morphisms from $T\Sigma$ to E. We say that (X_1, j_1) and (X_2, j_2) are *homotopic* if there exists an algebroid morphism $(X_{12}, j_{12}) : T(\Sigma \times I) \to E$ such that $(X_{12}, j_{12})(-, 0) = (X_1, j_1)$ and $(X_{12}, j_{12})(-, 1) = (X_2, j_2)$.

Next we show that the groupoid homotopies are the gauge transformations connected to the identities, i.e. those transformations that live in the component $(\mathcal{G}^{\Sigma}_X)_o$ of the source fibre $\mathcal{G}^{\Sigma}_X$ over $X : \Sigma \to \mathcal{M}$ connected to the identity X. Borrowing the terminology from gauge theory we say that groupoid homotopies are the small gauge transformations of $\mathcal{G}^{\Sigma}$.

Lemma 4.3. *Two groupoid morphisms $\hat{X}_i : \Pi(\Sigma) \to \mathcal{G}(E)$, $i = 1, 2$, are homotopic if and only if there exists a gauge transformation $\hat{\Phi} \in (\mathcal{G}^{\Sigma}_{X_1})_o$ such that $\hat{X}_2 = (\hat{X}_1)_{\hat{\Phi}}$.*

Proof. Let $\hat{X}_i$ be homotopic with homotopy $\hat{X}_{12}$. We have that

$$\hat{X}_2[c_{uv}] = \hat{X}_{12}([c_{uv}], 1, 1) = \hat{X}_{12}([c^{tr}_{uu}], 1, 0)\hat{X}_{12}([c_{uv}], 0, 0)\hat{X}_{12}([c^{tr}_{vv}], 0, 1)$$
$$= \hat{\Phi}(u)\hat{X}_1([c_{uv}])\hat{\Phi}(v)^{-1},$$

where $[c^{tr}_{uu}]$ is the class of the trivial loop through u, $\hat{\Phi} \in \mathcal{G}^{\Sigma}$ is defined by $\hat{\Phi}(u) = \hat{X}_{12}([c^{tr}_{uu}], 1, 0)$ and $\hat{\gamma}(s)(u) = \hat{X}_{12}([c^{tr}_{uu}], s, 0) \in \mathcal{G}(E)_{X_1(u)}$ is such that $\hat{\gamma}(0) = X_1$ and $\hat{\gamma}(1) = \hat{\Phi}$.

Conversely, let $\hat{\Phi} \in (\mathcal{G}^{\Sigma}_{X_1})_o$, with $\hat{\gamma} : I \to \mathcal{G}^{\Sigma}_{X_1}$ such that $\hat{\gamma}(0) = X_1$ and $\hat{\gamma}(1) = \hat{\Phi}$, and $\hat{X}_2 = (\hat{X}_1)_{\Phi}$. The required homotopy is then defined as $\hat{X}_{12}([c_{uv}], s_1, s_2) = \hat{\gamma}(s_1)(u)\hat{X}_1[c_{uv}]\hat{\gamma}(s_2)(v)^{-1}$. ∎

Below for completeness we describe the action of $\mathcal{G}^{\Sigma}$ on algebroid morphisms.

4.2. Groupoid action on the algebroid morphisms

In this subsection we compute the groupoid action of $\mathcal{G}^{\Sigma}$ on the algebroid morphism $(X, j) : T\Sigma \to E$. Let $\hat{\Phi} \in \mathcal{G}^{\Sigma}$ be such that $s(\hat{\Phi}) = X$ and let $\hat{X} : \Pi(\Sigma) \to \mathcal{G}(E)$ be the groupoid morphism integrating (X, j), e.g. $j = \hat{X}_*$. We define the action of $\hat{\Phi}$ on j as $j_{\hat{\Phi}} = \hat{X}_{\hat{\Phi}*}$.

Let $\mathcal{G}(E)^{(2)}_x = \{(\gamma_1, \gamma_2) \in \mathcal{G}(E) \times \mathcal{G}(E)_x \mid s(\gamma_1) = t(\gamma_2)\}$ and let $m : \mathcal{G}(E)^{(2)}_x \to \mathcal{G}(E)_x$ be the multiplication $m(\gamma_1, \gamma_2) = \gamma_1\gamma_2$. For each $v \in \Sigma$ we have that $\hat{X}_{\hat{\Phi}} : \Pi(\Sigma)_v \to \mathcal{G}(E)_{t(\hat{\Phi}(v))}$ can be expressed as $\hat{X}_{\hat{\Phi}} = R_{\hat{\Phi}(v)^{-1}} \circ m \circ (\hat{\Phi} \circ t, \hat{X})$, where $(\hat{\Phi} \circ t, \hat{X}) : \Pi(\Sigma)_v \to \mathcal{G}(E)^{(2)}_{X(v)}$, and $R_{\hat{\Phi}(v)^{-1}} : \mathcal{G}(E)_{X(v)} \to \mathcal{G}(E)_{t(\hat{\Phi}(v))}$ denotes the right multiplication by $\hat{\Phi}(v)^{-1}$. The tangent map $j_{\hat{\Phi}} : T_v\Sigma \to E_{t(\hat{\Phi}(v))}$ is expressed on $w \in T_v\Sigma$ as

$$j_{\hat{\Phi}}(w) = R_{\hat{\Phi}(v)^{-1}*} \circ m_*(\hat{\Phi}_*(w) \oplus j(w)). \tag{4.30}$$

Remark that in (4.30) we consider $\hat{\Phi}_*(w) \oplus j(w) \in T_{(\hat{\Phi}(v), X(v))}\mathcal{G}(E)^{(2)}_{X(v)} = \mathrm{Ker}(s_* - t_*) \subset T_{\hat{\Phi}(v)}\mathcal{G}(E) \oplus T_{X(v)}\mathcal{G}(E)_{X(v)}$, where $s_* - t_* : T_{\gamma_1}\mathcal{G}(E) \oplus T_{\gamma_2}\mathcal{G}(E)_x \to T_{s(\gamma_1)}M$ is defined on every $(\gamma_1, \gamma_2) \in \mathcal{G}(E)^{(2)}_{s(\gamma_1)}$. It is then clear that (4.30) makes sense only if $s_* \circ \hat{\Phi}_*(w) = t_* \circ j(w)$, e.g. $X_* = \rho \circ j$ which is (3.17). So the action of $\mathcal{G}^{\Sigma}$ automatically extends only to those vector bundle morphisms that commute with the anchor maps. The correct definition of the off shell gauge transformations is delicate and is beyond the scope of the present work, e.g. see [3] for a discussion of this problem.

Indeed the above construction is a direct generalization of the following example.

Example 4.4. Consider the Examples 2.2 and 2.11. Let E be the Lie algebra $\mathbf{g}$; then $\mathcal{G}(E) = \mathcal{G}(E)^{(2)} = G$ and $m_* = R_{g_2*} + L_{g_1*} : T_{g_1} \oplus T_{g_2} \to T_{g_1g_2}$. If we plug it in (4.30) we get the action of $g \in G^{\Sigma}$ on $j : T_v\Sigma \to \mathbf{g}$ as

$$j_g(w) = (R_{g(v)^{-1}} * \circ g_* + \mathrm{Ad}_{g(v)*}) \circ j(w). \tag{4.31}$$

In this example a Lie algebroid morphism is a flat connection on the trivial bundle and (4.31) is the gauge transformation of a connection. The associated groupoid morphism is defined by the parallel transport which transforms with the adjoint, accordingly to (4.29).

4.3. The open case

Let Σ be a surface with n boundary components, $\partial\Sigma = \bigcup_{i=1}^{n} \partial_i\Sigma$. According to the discussion in Section 3, we consider a set $\mathcal{E} = \{E_i\}$ of n subalgebroids $E_i \subset E$ over $\mathcal{D}_i$. Due to the integrability of E, there are n source-connected immersed Lie subgroupoids $\mathcal{G}(E_i) \subset \mathcal{G}(E)$ that integrate the Lie subalgebroids E_i.

We consider the space $\mathrm{Mor}(\Pi(\Sigma), \mathcal{G}(E); \mathcal{G}(\mathcal{E}))$ of groupoid morphisms $\hat{X} : \Pi(\Sigma) \to \mathcal{G}(E)$ such that $\hat{X}\left(\Pi(\Sigma)|_{\partial_i\Sigma}\right) \subset \mathcal{G}(E_i)$, where $\Pi(\Sigma)|_{\partial_i\Sigma} = \{[c_{uv}] \in \Pi(\Sigma) \mid c_{uv} \subset \partial_i\Sigma\}$ is the subgroupoid of $\Pi(\Sigma)$ integrating the i-th boundary component $\partial_i\Sigma$.

In analogy with the closed case, we define the groupoid $\mathcal{G}^{\Sigma,\mathcal{E}}$ over $\mathcal{M}^{\Sigma,\mathcal{E}}$, where

$$\mathcal{G}^{\Sigma,\mathcal{E}} = \{\hat{\Phi}\colon \Sigma \to \mathcal{G}(E) \mid \hat{\Phi}(\partial_i\Sigma) \subset \mathcal{G}(E_i)\}, \mathcal{M}^{\Sigma,\mathcal{E}} = \{\Phi\colon \Sigma \to \mathcal{M} \mid \Phi(\partial_i\Sigma) \subset \mathcal{D}_i\}.$$

Formula (4.29) gives a groupoid action of $\mathcal{G}^{\Sigma,\mathcal{E}}$ on $\mathrm{Mor}(\Pi(\Sigma), \mathcal{G}(\mathcal{M}); \mathcal{G}(\mathcal{E}))$.

From Sections 4.1 and 4.2 we can generalize all results for the case $\partial\Sigma \neq \emptyset$. The generalizations are rather straightforward. Thus we can introduce groupoid homotopies respecting the boundary conditions.

Definition 4.5. The groupoid morphisms $\hat{X}_i \in \mathrm{Mor}(\Pi(\Sigma), \mathcal{G}(E); \mathcal{G}(\mathcal{E}))$, $i = 1, 2$, are homotopic if there exists a groupoid morphism $\hat{X}_{12} : \Pi(\Sigma) \times I \times I \to \mathcal{G}(E)$ such that $\hat{X}_{12}(-, 0, 0) = \hat{X}_1$, $\hat{X}_{12}(-, 1, 1) = \hat{X}_{12}$ and $\hat{X}_{12} : \Pi(\Sigma)|_{\partial_i\Sigma} \times I \times I \to \mathcal{G}(E_i)$.

In analogy to the closed case, groupoid homotopies are the gauge transformations connected to the identities. In fact, let $(\mathcal{G}_X^{\Sigma,\mathcal{E}})_o$ be the component connected to $X : \Sigma \to \mathcal{M}$ of the source fiber over X. By repeating the same proof as in Lemma 4.3 and taking care of boundary conditions we can prove the following result.

Lemma 4.6. *Two groupoid morphisms* $\hat{X}_i \in \mathrm{Mor}(\Pi(\Sigma), \mathcal{G}(E); \mathcal{G}(\mathcal{E}))$, $i = 1, 2$, *are homotopic if and only if there exists a gauge transformation* $\hat{\Phi} \in (\mathcal{G}_{X_1}^{\Sigma,\mathcal{E}})_o$ *such that* $\hat{X}_2 = (\hat{X}_1)_{\hat{\Phi}}$.

5. Moduli space of solutions

In this section we discuss the moduli space of solutions modulo gauge transformations. As explained in the previous section, among several choices for the finite gauge transformations, we will choose the largest set, the groupoid $\mathcal{G}^{\Sigma}$. The main motivation for this choice comes from the PSM on the disk. In [4] it has been shown that this is the correct gauge group for the observables that are relevant for deformation quantization. Moreover the discussion of the moduli space is particularly simple and is the straightforward generalization of that obtained in [5] for Poisson–Lie groups.

As we will see, the whole construction is a quite direct generalization of the moduli space of flat G-connections. Let us analyze this case first. The algebroid morphisms from $T\Sigma$ to $\mathbf{g}$ correspond to the flat connections in the trivial G-bundle over Σ, where G is the simply connected Lie group integrating $\mathbf{g}$. The meaning of the second Lie theorem is that we can equivalently describe any flat connection for the trivial bundle by assigning the parallel transport. If $\dim \Sigma \leq 2$ then there are no other topologically inequivalent G-bundles and therefore the moduli space of algebroid morphisms coincides with the moduli space of flat connections; in generic dimension it will be the component corresponding to the trivial bundle. We are going to show that this description remains valid if we consider a generic integrable algebroid E and take the (ssc) groupoid $G(E)$ integrating it.

5.1. The closed case

Let Σ be a closed surface. We denote by $\mathcal{M}(\Sigma, \mathcal{G}(E))$ the space of groupoid morphisms divided by the action (4.29) of $\mathcal{G}^{\Sigma}$, i.e.

$$\mathcal{M}(\Sigma, \mathcal{G}(E)) = \mathrm{Mor}(\Pi(\Sigma), \mathcal{G}(E))/\mathcal{G}^{\Sigma}. \tag{5.32}$$

It is clear that, for any $(X, \hat{X}) \in \mathrm{Mor}(\Pi(\Sigma), \mathcal{G}(E))$, $X(\Sigma)$ is contained in a single leaf L. Since gauge transformations do not change the leaf, we can decompose the total moduli space (5.32) in the union of the moduli space $\mathcal{M}(\Sigma, \mathcal{G}(E); L)$ of solutions corresponding to the leaf L.

The following moduli space will be of central interest to us. Let us fix a leaf $L \subset \mathcal{M}$ and consider for any point $x_0 \in L$ the isotropy group $\mathcal{G}_{x_0}^{x_0} = \mathcal{G}(E)_{x_0}^{x_0}$ together with the moduli space of flat connections $F(\Sigma, \mathcal{G}_{x_0}^{x_0}) = \mathrm{Hom}(\pi_1(\Sigma), \mathcal{G}_{x_0}^{x_0})/\mathrm{Ad}$, where Ad denotes the adjoint action of $\mathcal{G}_{x_0}^{x_0}$. By varying $x_0 \in L$, the isotropy group changes by conjugation so that the moduli spaces are isomorphic; we denote it $F(\Sigma, \mathcal{G}(E); L)$. We define the moduli space of generalized flat $\mathcal{G}$-connections on Σ as the union over all symplectic leaves, e.g. $F(\Sigma, \mathcal{G}(E)) = \bigcup_L F(\Sigma, \mathcal{G}(E); L)$. It is important to notice that the moduli space can be introduced without referring to any choice of $x_0 \in L$. In fact one can verify that

$$F(\Sigma, \mathcal{G}(E)) = \bigcup_{y \in \mathcal{M}} \mathrm{Hom}(\pi_1(\Sigma), \mathcal{G}_y^y)/\mathrm{Ad}, \tag{5.33}$$

where Ad means the adjoint action of $\mathcal{G}(E)$, $\phi \to \gamma\phi\gamma^{-1}$, for $\phi \in \mathrm{Hom}(\pi_1(\Sigma), \mathcal{G}_y^y)$ and $\gamma \in \mathcal{G}_y$. Equivalently $F(\Sigma, \mathcal{G}(E)) = \mathrm{Mor}(\pi_1(\Sigma), \mathcal{G}(E))/\mathrm{Ad}$ is the moduli space of groupoid morphisms from $\pi_1(\Sigma)$ to $\mathcal{G}(E)$, where $\pi_1(\Sigma)$ is regarded as a groupoid over a point.

Remark 5.1. There is a very natural topology on $F(\Sigma, \mathcal{G}(E))$. Let Σ be the compact surface of genus g, then $\pi_1(\Sigma)$ is generated by $\{a_i, b_i\}_{i=1}^g$ with relation $\Pi_i[a_i, b_i] = 1$, where $[a, b] = aba^{-1}b^{-1}$. Let us define $\mathcal{G}_2^2 = \bigcup_{x \in \mathcal{M}} \times^{2g} \mathcal{G}(E)_x^x \subset \times^{2g} \mathcal{G}(E)$ together with the map $p : \mathcal{G}_2^2 \to \mathcal{G}(E)$,

$$p(z_1, w_1, \ldots, z_g, w_g) = [z_1, w_1] \ldots [z_g, w_g].$$

It is clear that $p^{-1}(\mathcal{M}) \subset \times^{2g} \mathcal{G}(E)$ inherits the relative topology and that $F(\Sigma, \mathcal{G}(E)) = p^{-1}(\mathcal{M})/\mathrm{Ad}$ the quotient topology. ∎

In Proposition 5.2 we show that the two moduli spaces (5.32) and (5.33) coincide. Before proving it, we will introduce the following auxiliary constructions. Let us fix $u_0 \in \Sigma$; we identify $\Pi(\Sigma)_{u_0}^{u_0}$ as $\pi_1(\Sigma)$ and $\Pi(\Sigma)_{u_0}$ as $\tilde{\Sigma}$, the universal cover of Σ. Let us introduce the following trivialization for the $\pi_1(\Sigma)$ principal bundle $\tilde{\Sigma} \to \Sigma$. Let $\{U_\alpha\}_\alpha$ be a covering of Σ with U_α and $U_\alpha \cap U_\beta$ contractible. Let us fix for each $u \in U_\alpha$ a curve $c_{u_0 u}^\alpha$ starting in u and ending in u_0 in such a way that, once U_α is contracted, all such curves are homotopic. Then let us define on $U_\alpha \cap U_\beta$, $h_{\alpha\beta} = c_{u_0 u}^\alpha \circ c_{u u_0}^\beta$; it is clear that $[h_{\alpha\beta}] \in \pi_1(\Sigma)$ is constant for all $u \in U_\alpha \cap U_\beta$; we have defined a flat structure on $\tilde{\Sigma}$.

Proposition 5.2. *For every closed manifold Σ such that* $\dim \Sigma = 1, 2$ *and for every source simply connected Lie groupoid $\mathcal{G}(E)$ we have $\mathcal{M}(\Sigma, \mathcal{G}(E)) = F(\Sigma, \mathcal{G}(E))$.*

Proof. For any $(X, \hat{X}) \in \mathrm{Mor}(\Pi(\Sigma), \mathcal{G}(E))$ we have that $\hat{X} : \pi_1(\Sigma) \to \mathcal{G}_{X(u_0)}^{X(u_0)}$ is a group homomorphism. If we change $(X, \hat{X})$ by a gauge transformation $\hat{\Psi} \in \mathcal{G}^\Sigma$ we get that $\hat{X}_{\hat{\Psi}}|_{\pi_1(\Sigma)} = \mathrm{Ad}_{\hat{\Psi}(u_0)}(\hat{X}|_{\pi_1(\Sigma)})$, so that we associate an element in $F(\Sigma, \mathcal{G}(E))$ to the class in $\mathcal{M}(\Sigma, \mathcal{G}(E))$ represented by $(X, \hat{X})$.

Let us show that this correspondence is injective. Let $(X_i, \hat{X}_i)$ be two solutions corresponding to the same flat connection, *e.g.* $\mathrm{Ad}_{\gamma_{21}}(\hat{X}_1|_{\pi_1(\Sigma)}) = \hat{X}_2|_{\pi_1(\Sigma)}$, for some $\gamma_{21} \in \mathcal{G}(E)_{X_1(u_0)}^{X_2(u_0)}$. If we introduce the local lifting $\psi_{i\alpha} : U_\alpha \to \mathcal{G}_{X_i(u_0)}$, $\psi_{i\alpha}(u) = \hat{X}_i[c_{uu_0}^\alpha]$, it is easy to verify that $\hat{\Phi}_\alpha(u) = \psi_{2\alpha}(u)\gamma_{21}\psi_{1\alpha}^{-1}(u)$ for $u \in U_\alpha$ extends to a globally defined map $\hat{\Phi} \in \mathcal{G}^\Sigma$ such that $s(\hat{\Phi}) = X_1$, $t(\hat{\Phi}) = X_2$ and $\hat{X}_2 = (\hat{X}_1)_{\hat{\Phi}}$.

Let us go in the opposite direction and show that the correspondence is surjective. In order to do this we first recall that a G-bundle $E \to B$ is n-universal if $\pi_i(E) = 0$ for $i < n$. The following facts are relevant for us: every G-bundle over an n-dimensional manifold N is the pull-back of E for some $X : N \to B$; moreover if E is $n+1$-universal, then the G-bundles over N are classified by homotopies from N to B [23].

Now let $\rho : \pi_1(\Sigma) \to \mathcal{G}_{x_0}^{x_0}$ be a flat connection for some $x_0 \in L$; since $\mathcal{G}(E)_{x_0}$ is simply connected, then $\mathcal{G}_{x_0} \to L_{x_0}$ is a universal 2-bundle for $\mathcal{G}_{x_0}^{x_0}$. This means that the principal $\mathcal{G}_{x_0}^{x_0}$-bundle $\tilde{\Sigma} \times_\rho \mathcal{G}_{x_0}^{x_0}$ is equivalent to the pull-back $X_\rho^*(\mathcal{G}_{x_0})$ for some $X_\rho : \Sigma \to L_{x_0}$. Let $\Psi_\rho : \tilde{\Sigma} \times_\rho \mathcal{G}_{x_0}^{x_0} \to \mathcal{G}_{x_0}$ be the bundle map. Finally, define $\hat{X}_\rho : \pi(\Sigma) \to \mathcal{G}(E)$ as

$$\hat{X}_\rho[c_{uv}] = \Psi_\rho([c_{uu_0}^\alpha], e)\rho[c_{u_0 u}^\alpha c_{uv} c_{vu_0}^\beta]\Psi_\rho([c_{vu_0}^\beta], e)^{-1} u \in U_\alpha, v \in U_\beta.$$

It is easy to see that $(X_\rho, \hat{X}_\rho)$ is a well defined groupoid morphism. ∎

We remark the crucial role played in the proof by the fact that $\mathcal{G}_{x_0}$ is simply connected which follows from the assumption that $\mathcal{G}(E)$ is source simply connected. Indeed the same assumption was used for the Poisson–Lie group case, see [5]. If the groupoid $\mathcal{G}(E)$ is not source simply connected or $\dim \Sigma > 2$ then the Proposition 5.2 is not true anymore and we only have the embedding $\mathcal{M}(\Sigma, \mathcal{G}(E)) \subset F(\Sigma, \mathcal{G}(E))$.

Remark 5.3. We can rephrase the above construction by saying that to every groupoid morphism $(X, \hat{X})$ we can associate a flat $\mathcal{G}_{x_0}^{x_0}$-bundle over Σ. In order to be more explicit, we are going to show that the pull-back principal bundle $X^*(\mathcal{G}_{x_0})$ admits a flat structure, i.e. according to [15] it is isomorphic to $\tilde{\Sigma} \times_{\hat{X}} \mathcal{G}_{x_0}^{x_0}$. In fact it is straightforward to verify that $\Phi_X : X^*(\mathcal{G}(E)_{x_0}) \to \tilde{\Sigma} \times_{\hat{X}} \mathcal{G}_{x_0}^{x_0}$ defined by

$$\Phi_X(u, \gamma) = ([c^\alpha_{uu_0}], \hat{X}[c^\alpha_{u_0 u}]\gamma), \quad \Phi_X^{-1}([c_{uu_0}], \gamma) = (u, \hat{X}[c_{uu_0}]\gamma)$$

is a well defined principal bundle isomorphism.

The following examples will help to clarify the above constructions.

Example 5.4. Consider Example 4.4 and let $\mathcal{G} = G$ be a simply connected Lie group seen as a groupoid over a point $*$; then $\mathcal{G}_* = \mathcal{G}_*^* = G$. Then G^Σ can be seen as a groupoid over a point, hence a group, and $\mathrm{Bis}(G^\Sigma) = G^\Sigma$. The moduli space of solutions coincides with the moduli space of flat G-connections on Σ divided by gauge transformations.

Example 5.5. There are two extreme cases where the moduli spaces of solutions are easy to describe. The first one is when $\mathcal{M}$ is a simply connected symplectic manifold. In fact the (ssc) groupoid integrating it is $\mathcal{G}(\mathcal{M}) = \mathcal{M} \times \mathcal{M}$ and $\mathcal{G}_{x_0}^{x_0} = (x_0, x_0)$; the space of flat connection $F(\Sigma, \mathcal{G}_{x_0}^{x_0}) = \{*\}$ is then trivial. The other one is when $\Sigma = \mathbf{S}^2$ where we have that $\mathcal{M}(\mathbf{S}^2, \mathcal{G}(E))$ is the space of leaves of $\mathcal{G}(E)$.

Example 5.6. Let $\mathcal{G} = \mathcal{M} \times G$ be the action groupoid. Then a leaf is an orbit $L_{x_0} = G/G_{x_0}$, for $x_0 \in M$. We have that $\mathcal{G}^\Sigma = \mathcal{M}^\Sigma \times G^\Sigma$ and thus it is enough to consider G^Σ as a gauge group since the orbits of $\mathcal{G}^\Sigma$ and G^Σ coincide. In fact, the actions of $\hat{\Psi} = (\psi, \gamma) \in \mathcal{G}^\Sigma$ and $\gamma \in G^\Sigma$ on $\hat{X}$ coincide. This was the gauge group considered in [5].

5.2. The open case

Let Σ be a compact surface with boundary. Let us consider the case with one boundary component $\partial\Sigma$. Let $E' \to \mathcal{D}$ be a subalgebroid of E and let $\mathcal{G}(E')$ be the (source connected) immersed subgroupoid of $\mathcal{G}(E)$ integrating it. We define the relevant moduli space as the space of groupoid morphisms $\mathrm{Mor}(\Pi(\Sigma), \mathcal{G}(E); \mathcal{G}(E'))$ respecting the boundary conditions defined in Section 4.3 divided by the action (4.29) of $\mathcal{G}^{\Sigma, E'}$, e.g.

$$\mathcal{M}(\Sigma, \mathcal{G}(E); \mathcal{G}(E')) = \mathrm{Mor}(\Pi(\Sigma), \mathcal{G}(E); \mathcal{G}(E'))/\mathcal{G}^{\Sigma, E'}. \tag{5.34}$$

It is clear that each solution sends the boundary in a fixed leaf $L \subset \mathcal{D}$ of E' and we denote with $\mathcal{M}(\Sigma, \mathcal{G}(E); \mathcal{G}(E'), L)$ the subset of (5.34) corresponding to this leaf.

In analogy to the closed case, we introduce the following moduli space of generalized flat connections with the holonomy around the boundary which takes value in a subgroup. The relevant space is the union over the leaves $L_x \subset \mathcal{D}$ of $\mathcal{G}(E')$ of the moduli spaces of flat $\mathcal{G}(E)_x^x$-connections with holonomy around the boundary living in $\mathcal{G}(E')_x^x$. More precisely, let us choose $u_0 \in \partial\Sigma$ and identify $\pi_1(\Sigma) = \Pi(\Sigma)_{u_0}^{u_0}$ and denote with $[\partial\Sigma] \in \pi_1(\Sigma)$ the boundary generator. We define

$$F(\Sigma, \mathcal{G}(E); \mathcal{G}(E')) = \bigcup_{x \in \mathcal{D}} \{\rho \in \mathrm{Mor}(\pi_1(\Sigma), \mathcal{G}(E)_x^x),\ \rho[\partial\Sigma] \in \mathcal{G}(E_x'^x)\}/\mathrm{Ad}, \tag{5.35}$$

where Ad is the adjoint action of $\mathcal{G}(E')$.

Let $\{U_\alpha, c^\alpha_{uu_0}\}$ define a trivialization of $\tilde{\Sigma} = \Pi(\Sigma)_{u_0}$ as described in Section 5.1 such that $c^\alpha_{uu_0} \subset \partial\Sigma$ for all $u \in \partial\Sigma$.

Proposition 5.7. *For every surface Σ ($\dim \Sigma \leq 2$) with one boundary component and every Lie source simply connected groupoid $\mathcal{G}(E)$ and immersed source connected Lie subgroupoid $\mathcal{G}(E') \subset \mathcal{G}(E)$ we have that $F(\Sigma, \mathcal{G}(E); \mathcal{G}(E')) = \mathcal{M}(\Sigma, \mathcal{G}(E); \mathcal{G}(E'))$.*

Proof. The proof consists in repeating the same steps of the proof of Proposition 5.2 and checking that the boundary conditions are respected. The map from $\mathcal{M}(\Sigma, \mathcal{G}(E); \mathcal{G}(E'))$ to $F(\Sigma, \mathcal{G}(E); \mathcal{G}(E'))$ is defined in the same way and easily shown to be injective, thanks to the choice of trivialization of $\tilde{\Sigma}$. More care is needed for the inverse map. Let $\rho : \pi_1(\Sigma) \to \mathcal{G}(E)_x^x$ with $\rho[\partial\Sigma] \in \mathcal{G}(E')_x^x$, for $x \in \mathcal{D}$. We can then define the $\mathcal{G}(E)_x^x$-bundle $\tilde{\Sigma} \times_\rho \mathcal{G}(E)_x^x$ over Σ and the $\mathcal{G}(E')_x^x$-bundle $\partial\tilde{\Sigma} \times_\rho \mathcal{G}(E')_x^x$ over $\partial\Sigma$. Since $\mathcal{G}(E')$ is source connected, the $\mathcal{G}(E')_x^x$ bundle $t : \mathcal{G}(E')_x \to L'_x$, where $L'_x \subset \mathcal{D}$ is the leaf containing x, is 1-universal. Then there exists a $\mathcal{G}(E')_x^x$ bundle map $\psi'_\rho : \partial\tilde{\Sigma} \times_\rho \mathcal{G}(E')_x^x \to \mathcal{G}(E')_x$ that can be extended to a $\mathcal{G}(E)_x^x$ bundle map $\psi_\rho : \tilde{\Sigma} \times_\rho \mathcal{G}(E)_x^x|_{\partial\Sigma} \to \mathcal{G}(E)_x$. Since $\mathcal{G}(E)_x \to L_x$ is 2-universal for $\mathcal{G}(E)_x^x$, we have that ψ_ρ can be extended to the whole bundle over Σ by $\Psi_\rho : \Sigma \times_\rho \mathcal{G}(E)_x^x \to \mathcal{G}(E)_x$. The groupoid morphisms is defined then as in the proof of Proposition 5.2 and respects the boundary conditions. ■

Example 5.8. Let us consider $\Sigma = D^1$. Then $\mathcal{M}(\Sigma, \mathcal{G}(E); \mathcal{G}(E'))$ is the space of leaves of $\mathcal{G}(E')$.

We close this section with a few remarks regarding the moduli space over the interval $I = [0, 1]$. This is closely related to the explicit construction of the groupoid $\mathcal{G}(E)$ integrating the Lie algebroid done in [12] and [14]. We consider first the Lie groupoid morphisms with boundary conditions given by the trivial groupoid over $\mathcal{M}$ (i.e., $E_0 = E_1 = \mathcal{M} \times \{0\}$ and $\mathcal{G}(E_0) = \mathcal{G}(E_1) = \mathcal{M}$). Remark that any groupoid morphism $\hat{X} : I \times I \to \mathcal{G}(E)$ satisfies these boundary conditions since they simply mean that $\hat{X}(0, 0), \hat{X}(1, 1) \in \mathcal{M}$. The gauge transformations $\hat{\Phi} \in \mathcal{G}^{I,\mathcal{M},\mathcal{M}}$ are given by maps $\hat{\Phi} : I \to \mathcal{G}(E)$ such that $\hat{\Phi}(0), \hat{\Phi}(1) \in \mathcal{M}$. Then using a standard argument, it is easy to see that the map $\hat{X} \to \hat{X}[1, 0]$ defines a bijection between the moduli space and the groupoid itself, i.e.

$$\mathrm{Mor}(I \times I, \mathcal{G}(E); \mathcal{M}, \mathcal{M})/\mathcal{G}^{I,\mathcal{M},\mathcal{M}} = \mathcal{G}(E).$$

This description of the groupoid must be compared with that of [12,14], where the groupoid is obtained as the space of algebroid morphisms divided by algebroid homotopies. By taking into account Lemma 4.6, it is reasonable to think that $\mathcal{G}^{I,\mathcal{M},\mathcal{M}}$ is connected to the identities.

The case of generic boundary conditions can be analogously treated. Let $\mathcal{E} = \{E_0, E_1\}$ be two subalgebroids of E and let $\mathcal{G}(E_i)$, $i = 0, 1$, be the two subgroupoids of $\mathcal{G}(E)$ integrating them. It is easy to see that

$$\mathrm{Mor}(I \times I, \mathcal{G}(E); \mathcal{G}(E_0), \mathcal{G}(E_1))/\mathcal{G}^{I,\mathcal{E}} = \mathcal{G}(E_1) \setminus t^{-1}(\mathcal{G}(E_1)) \cap s^{-1}(\mathcal{G}(E_0))/\mathcal{G}(E_0).$$

6. Concluding remarks

In this work we have studied the space of Lie algebroid (groupoid) morphisms modulo gauge transformations. Since our motivations come from two-dimensional topological field theory, the point of view has been gauge theoretic. In this perspective, we argued that the choice of finite gauge transformations as the transformations (4.29) of the groupoid $\mathcal{G}^\Sigma$ is the most natural one. Indeed the whole story is just a relatively direct generalization of the group case and the moduli spaces can be thought of as a generalization of the moduli spaces of flat connections.

Since groupoids are categories, it is extremely useful to reconsider the paper from a categorical point of view, where these choices appear as extremely natural. In fact, let $\mathcal{H}$ and $\mathcal{G}$ be two groupoids, then a groupoid morphism from $\mathcal{H}$ to $\mathcal{G}$ is a covariant functor. We can consider the functor category $\mathbf{C}(\mathcal{H}, \mathcal{G})$, whose objects are the groupoid morphisms $\mathrm{Mor}(\mathcal{H}, \mathcal{G})$ and whose morphisms are the natural transformations between functors; $\mathbf{C}(\mathcal{H}, \mathcal{G})$ is again a groupoid. It is easy to verify that the gauge transformations defined in (4.29) coincide with the natural transformations. So the moduli space defined in (5.32) corresponds to the set $\pi_0(\mathbf{C})$ of the connected components of $\mathbf{C}$ when $\mathcal{H} = \Pi(\Sigma)$ and $\mathcal{G} = \mathcal{G}(E)$. The content of Proposition 5.2 can be expressed by saying that when $\dim\Sigma \leq 2$ then $\pi_0(\mathbf{C}(\pi_1(\Sigma), \mathcal{G}(E))) = \pi_0(\mathbf{C}(\Pi(\Sigma), \mathcal{G}(E)))$.

Moreover, it is important to point out that in the closed case the moduli spaces are *Morita invariants* of the groupoid $\mathcal{G}(E)$ (see [7] for definitions). This fact follows from the observation that if $\mathcal{G}_1$ and $\mathcal{G}_2$ are Morita equivalent then also $\mathbf{C}(\pi_1(\Sigma), \mathcal{G}_i)$ are Morita equivalent groupoids: in particular they have the same space of connected components.

On a more geometrical side, it will be extremely interesting to see which geometrical structures can be defined over these moduli spaces. Indeed some basic facts can be observed now. In Remark 5.1 we pointed out that the moduli spaces are topological spaces. Moreover, they are the union of moduli spaces of flat connections and thus they are a collection of symplectic manifolds (with singularity).

We hope to come back to all these problems in the future and consider what the quantization of the TFT can bring to the understanding of these spaces. The TFT which one can associate to any Lie algebroid is a BF-like theory. In the group case the quantization of BF-theory gives rise to many interesting calculations, e.g. the Ray–Singer torsion. It will be interesting to see if those calculations can be extended to the general case of Lie groupoids.

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References

[1] M. Blau, G. Thompson, Topological Gauge theories of antisymmetric tensor fields, Ann. Phys. 205 (1991) 130.
[2] M. Blau, G. Thompson, Lectures on 2-d gauge theories: Topological aspects and path integral techniques, arXiv:hep-th/9310144.
[3] M. Bojowald, A. Kotov, T. Strobl, Lie algebroid morphisms, Poisson sigma models, and off-shell closed gauge symmetries, J. Geom. Phys. 54 (2005) 400. arXiv:math.dg/0406445.
[4] F. Bonechi, A. Cattaneo, M. Zabzine, Geometric quantization and non-perturbative Poisson sigma model, arXiv:math.SG/0507223.
[5] F. Bonechi, M. Zabzine, Poisson sigma model over group manifolds, J. Geom. Phys. 54 (2005) 173–196. arXiv:hep-th/0311213.
[6] P. Bressler, A. Chervov, Courant algebroids, J. Math. Sci. 128 (4) (2005) arXiv:hep-th/0212195.
[7] H. Bursztyn, A. Weinstein, Poisson geometry and Morita equivalence, arXiv:math.SG/0402347.
[8] I. Calvo, F. Falceto, Poisson reduction and branes in Poisson-Sigma models, Lett. Math. Phys. 70 (2004) 231–247.
[9] A. Cannas da Silva, A. Weinstein, Geometric Models for Noncommutative Algebras, in: Berkeley Mathematics Lecture Notes, AMS, Providence, 1999.
[10] A. Cattaneo, On the integration of Poisson manifolds, Lie algebroids, and co-isotropic submanifolds, Lett. Math. Phys. 67 (2004) 33–48. arXiv:math.SG/0308180.
[11] A.S. Cattaneo, G. Felder, A path integral approach to the Kontsevich quantization formula, Commun. Math. Phys. 212 (2000) 591–611. arXiv:math.qa/9902090.
[12] A.S. Cattaneo, G. Felder, Poisson sigma models and symplectic groupoids, in: N.P. Landsman, M. Pflaum, M. Schlichen-meier (Eds.), Quantization of Singular Quotients, in: Progress in Mathematics, vol. 198, 2001, p. 4173. arXiv:math.SG/0003023.
[13] A.S. Cattaneo, G. Felder, Coisotropic submanifolds in Poisson geometry and branes in the Poisson sigma model, Lett. Math. Phys. 69 (2004) 157–175. arXiv:math/0309180.
[14] M. Crainic, R.L. Fernandes, Integrability of Lie brackets, Ann. of Math. (2) 157 (2) (2003) 575–620. arXiv:math.DG/0105033.
[15] P. De Bartolomeis, Principal bundles in action, Riv. Mat. Univ Parma (4) 17 (1991) 1–65.
[16] P.J. Higgins, K. Mackenzie, Algebraic constructions in the category of Lie algebroids, J. Algebra 129 (1) (1990) 194–230.
[17] G.T. Horowitz, Exactly Soluble Diffeomorphism Invariant Theories, Commun. Math. Phys. 125 (1989) 417.
[18] N. Ikeda, Two-dimensional gravity and nonlinear gauge theory, Ann. Phys. 235 (1994) 435. arXiv:hep-th/9312059.
[19] J.H. Lu, A. Weinstein, Groupoides symplectiques doubles de groupes de Lie-Poisson, in: Comptes Rendus de Seances, Academie de Sciences (Paris), Serie I. Mathematique, vol. 309, 1989, pp. 951–954.
[20] K.C.H. Mackenzie, General Theory of Lie Groupoids and Lie Algebroids, Cambridge University Press, Cambridge, 2005, pp. xxxviii+501.
[21] P. Schaller, T. Strobl, Poisson structure induced (topological) field theories, Mod. Phys. Lett. A 9 (1994) 3129. arXiv:hep-th/9405110.
[22] T. Strobl, Algebroid Yang-Mills Theory, Phys. Rev. Lett. 93 (2004) 211601. arXiv:hep-th/0406215.
[23] N. Steenrod, The Topology of Fibre Bundles, Princeton University Press, 1951.